

FLOW EQUIVALENCE AND ISOTOPY FOR SUBSHIFTS

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ABSTRACT. We study basic properties of flow equivalence on one-dimensional compact metric spaces with a particular emphasis on isotopy in the group of (self-) flow equivalences on such a space. In particular, we show that an orbit-preserving such map is not always an isotopy, but that this always is the case for suspension flows of irreducible shifts of finite type. We also provide a version of the fundamental discretization result of Parry and Sullivan which does not require that the flow maps are either injective or surjective. Our work is motivated by applications in the classification theory of sofic shift spaces, but has been formulated to supply a solid and accessible foundation for other purposes.

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1. INTRODUCTION

In this paper we set out some basic results around flow equivalence and isotopy involving flows without fixed points on one-dimensional compact metric spaces and the return maps to their cross sections. Our motivation is to provide a solid and accessible foundation for other work.

The study of flow equivalence of shifts of finite type (SFTs) is very well understood and has had profound applications to C^* -algebras. We use several results in our work on flow equivalence within certain

subshift classes [7, 8]. Flow equivalence of G shifts of finite type has provided some information about knot invariants.

We are also interested in the “mapping class group” of a subshift, especially of an irreducible shift of finite type. This is the group of self-homeomorphisms of the mapping torus of that subshift up to isotopy, studied in [6, 13].

By a flow map, we mean a continuous map between spaces with flows, which maps each domain orbit onto some range orbit by an orientation preserving local homeomorphism. In Section 2, we give basic background on flows and cross sections. In Section 3 we study when a flow equivalence mapping each orbit into itself is isotopic to the identity within the group of flow equivalences of a space Y to itself. Even for subshifts, this can be tricky. Suppose Y is the mapping torus of a subshift X and $f : Y \rightarrow Y$ is a flow equivalence mapping each orbit into itself. Must f be isotopic to the identity? The answer is yes if X is a minimal shift [2, Theorem 2.5] or if X is an irreducible shift of finite type (Theorem 6.4); but for a reducible shift of finite type or mixing sofic shift, the answer is no (Examples 3.4, 3.5). The main criterion for this isotopic triviality is given in Theorem 3.3; it should be known, but we haven’t found its statement in the literature, despite the abundance of related results.

In Section 4, we give a formulation and extension of the key argument of the Parry-Sullivan paper [26] which is the basis for connecting the dynamics of one-dimensional flows and the discrete systems given by return maps to cross sections. In particular, we give a version applicable to flow maps which are neither surjective nor injective (which we need in [8] to study flow equivalence of sofic shifts via their canonical SFTs covers). In Section 5, we introduce flow codes, which play for flow equivalence of subshifts the role block codes play for homomorphisms of subshifts.

In Section 6, given a flow equivalence of irreducible SFTs respecting lengths of finite orbits, we show it is induced by a conjugacy, and use this to prove Theorem 6.4. Also, given a flow equivalence of systems Y, Y' with cross sections C, C' , we characterize when a flow equivalence $Y \rightarrow Y'$ can be lifted to an equivalence $C \times \mathbb{R} \rightarrow C' \times \mathbb{R}$ of their covering spaces.

In Section 7, for flows on one-dimensional spaces we prove two extension results. An isotopically trivial map on a subflow can be extended to an isotopically trivial map on the entire flow. A cross section of a subflow can be extended to a cross section of the entire flow.

Some results are a stripped down version of ideas and results from the theory of smooth flows on hyperbolic sets (see [20, Sections 2.2, 2.9 and 19.2]), as we indicate. We have given independent proofs for these results, because the the smooth statements don’t include the zero

dimensional case; some smooth arguments do not translate mechanically to the zero dimensional setting; and some arguments adequate for dimension zero are much shorter and easier.

The one-dimensional spaces Y we study can be considered as (a quite special class of) tiling spaces. The large literature on tiling spaces contains results (see e.g. in [19]) which specialize to imply some of our statements in cases, e.g. when the suspension flow on Y is minimal.

Notation. When L_1 and L_2 are two sets of words, then we let $L_1 L_2$ denote the set $\{uv : u \in L_1, v \in L_2\}$. When L is a set of words, then we let L^* denote the set of words that are concatenations of zero or more elements from L (so the empty word is an element of L^*).

Acknowledgements. We thank Sompong Chuysurichay for valuable comments on an earlier draft of this paper. This work was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92), and by VILLUM FONDEN through the network for Experimental Mathematics in Number Theory, Operator Algebras, and Topology.

2. FLOWS AND CROSS SECTIONS

In this section, we give basic background on flows and cross sections.

Definition 2.1. A *flow* in this paper is a continuous and *fixed point free*¹ action of \mathbb{R} on a compact metric² space. A flow on a compact metric space Y is given by a continuous map $\gamma : Y \times \mathbb{R} \rightarrow Y$ such that for all s, t in \mathbb{R} and y in Y , $\gamma(\gamma(y, s), t) = \gamma(y, s + t)$, and $\gamma(y, 0) = y$. For t in \mathbb{R} , $y \mapsto \gamma(y, t)$ defines the time t homeomorphism $\gamma_t : Y \rightarrow Y$.

Definitions 2.2. Given a compact metric space X , a homeomorphism $T : X \rightarrow X$, and a continuous function $f : X \rightarrow (0, \infty)$, let Y be the quotient of $\{(x, t) : 0 \leq t \leq f(x)\}$ by the identifications $(x, f(x)) \sim (T(x), 0)$ for all x in X . The map $\gamma_t : ((x, s), t) \mapsto (x, s + t)$ is a flow on $X \times \mathbb{R}$, which commutes with the \mathbb{Z} action generated by $(x, t) \mapsto (T(x), t - f(x))$. The space Y can be presented as the orbit space of this \mathbb{Z} action. The flow on $X \times \mathbb{R}$ induces a flow on Y . This is one construction of the *flow under a function*. This presentation generalizes to other groups.

The space X is the *base* and f is the *ceiling function*. In the case that f is the constant function 1, Y is the *mapping torus* of T , which (abusing notation) we denote SX . The induced flow on SX is the *suspension of T* .

¹For flows with fixed points, flow equivalence (as in Definitions 2.9) is a much weaker relation than for flows without fixed points (see e.g. [23]), and compactness arguments are much less effective.

²In this paper, the choice of metric compatible with the topology won't matter.

Definition 2.3. [31, Sec. 7] Let $\gamma : Y \times \mathbb{R} \rightarrow Y$ be a flow as in Definition 2.1. A *cross section* to the flow is a closed subset C of Y such that the restriction of γ to $C \times \mathbb{R} \rightarrow Y$ is a surjective local homeomorphism.

Remark 2.4. For a flow γ on Y , it is not difficult to check that the following are equivalent conditions on a closed subset C of Y .

- (1) C is a cross section to the flow.
- (2) $\gamma : C \times \mathbb{R} \rightarrow Y$ is surjective, and there exists $\delta > 0$ such that $\gamma : C \times (-\delta, \delta) \rightarrow \gamma(C \times (-\delta, \delta))$ is a local homeomorphism.
- (3) $\gamma : C \times \mathbb{R} \rightarrow Y$ is surjective, and there exists $\delta > 0$ such that $\gamma : C \times (-\delta, \delta) \rightarrow \gamma(C \times (-\delta, \delta))$ is a homeomorphism.
- (4) $\gamma : C \times \mathbb{R} \rightarrow Y$ is surjective, and there is a well defined continuous return time function $\tau_C : C \rightarrow \mathbb{R}$, given by $\tau_C(x) = \min\{t > 0 : \gamma_t(x) \in C\}$.
- (5) $\gamma : C \times \mathbb{R} \rightarrow Y$ is surjective, and there is a well defined return time function $\tau_C : C \rightarrow \mathbb{R}$, given by $\tau_C(x) = \min\{t > 0 : \gamma_t(x) \in C\}$, which is bounded away from 0.

For the flow under a function in Definitions 2.2, the image of $X \times \{0\}$ in Y is a cross section for the flow. Conversely, by Remark 2.4, if C is a cross section to a flow γ on Y , then the flow γ is topologically conjugate (as defined in Definition 2.9) to the flow under a function built with base C and ceiling function the return time function on C .

Large classes of flows will not admit a cross section (see e.g. [28, 31, 15]), but a flow on a one-dimensional space will always admit a cross section. This fact, stated in Proposition 2.8 below, is the analogue for continuous flows on compact one-dimensional metric spaces of the Ambrose-Kakutani Theorem [3, 4] for aperiodic measure preserving flows on a standard probability space.

We will be using the following basic tool for studying flows.

Theorem 2.5. [24, Thm. V.2.15] *Suppose $\gamma : Y \times \mathbb{R} \rightarrow Y$ is a flow on a compact metric space Y , $p \in Y$, $T > 0$ and $|t| \leq 4T \implies \gamma(p, t) \neq p$.*

Then there exists a closed set F in Y such that γ maps $F \times [-T, T]$ homeomorphically onto a neighborhood of p .

Remark 2.6. If F is a closed set in Y , J is an interval and γ is injective on $W = F \times J$ and $\gamma(W)$ has nonempty interior, then F is called a *local section* (or *local cross section*) and $\gamma(W)$ is called a *flowbox* (a *flowbox neighborhood for points in its interior*). In settings with more regularity (differentiable, Lipschitz, ...) more conditions might be demanded of the flow in the flowbox.

Remark 2.7. The statement of [24, Thm. V.2.15] does not quite cover the statement of Theorem 2.5. However, the proof of [24, Thm. V.2.15] finds for arbitrarily small $\tau > 0$ a closed set F in Y such that $\gamma(F \times [-2\tau, 2\tau])$ contains a neighborhood of p and γ is injective on

$F \times [-T, T]$. This neighborhood contains $\gamma(F \times [-(T - 4\tau), T - 4\tau])$. Beginning with some $T + \epsilon$ in place of T we get Theorem 2.5.

Proposition 2.8 below is well known, and is a special case of more general results [1, 21], but for completeness we will include a short proof. By “dimension”, we mean covering dimension (but for compact metric spaces, various standard conditions agree [17]). We let $B_\delta(y)$ denote the open ball with radius δ and center y .

Proposition 2.8. *Suppose γ is a flow on a one-dimensional compact metric space Y . Then the flow has a cross section, and every cross section of the flow is zero dimensional.*

Proof. Pick $T > 0$ with $4T$ smaller than the period of any point. Given $y \in Y$, let $U = \gamma(F \times [-T, T])$ be a flowbox neighborhood of y as in Theorem 2.5. Being an injective map between compact Hausdorff spaces, the restriction of γ to $F \times [-T, T]$ is a homeomorphism onto U . Because Y is one-dimensional, it follows that F is zero dimensional³. Take C relatively clopen in F such that $\gamma(C \times (-T, T))$ is an open neighborhood U of y .

Let $\{U_i\} = \{\gamma(C_i \times (-T, T)) : 1 \leq i \leq N\}$ be a finite collection of such sets whose union covers Y . Given $j \neq 1$ and p in $C_1 \cap C_j$, because U_j is open there will be a relative neighborhood $W_{p,j}$ of p in C_1 such that $W_{p,j} \subset U_j$. We take $W_{p,j}$ clopen in C_1 ; by compactness, a finite union W_j of such sets covers $C_1 \cap C_j$. Replace C_1 with $C_1 \setminus \bigcup_{j>1} W_j$. Now C_1 is disjoint from the other C_j ; for a small $T_1 > 0$, $\gamma(C_1 \times (-T_1, T_1))$ is still open in Y and is disjoint from the sets $\gamma(C_j \times (-T_1, T_1))$, and every orbit still intersects $\bigcup_i C_i$. Iterating this move, we find $\epsilon = T_N > 0$ and compact disjoint zero dimensional sets C_1, \dots, C_N such that the open sets $U_j = \gamma(C_j \times (-\epsilon, \epsilon))$ are disjoint and every orbit hits some C_j . Let $C = \bigcup_j C_j$. Then $\gamma : C \times \mathbb{R} \rightarrow Y$ is surjective and $\gamma : C \times (-\epsilon, \epsilon) \rightarrow Y$ is a homeomorphism. Therefore C is a cross section. \square

Definitions 2.9. A *homomorphism* of flows $(Y_1, \gamma_1) \rightarrow (Y_2, \gamma_2)$ is a continuous map $h : Y_1 \rightarrow Y_2$ such that $h(\gamma_1(y, t)) = \gamma_2(h(y), t)$ for all $t \in \mathbb{R}$ and all $y \in Y_1$. An *epimorphism* (or *semiconjugacy*) of flows is a surjective homomorphism of flows; an *isomorphism* (or *conjugacy* or *topological conjugacy*) of flows is a homomorphism of flows defined by a homeomorphism⁴.

A *flow map* is a continuous map $h : Y_1 \rightarrow Y_2$ such that for every y in Y_1 , the restriction of h to the γ_1 orbit of y is an orientation-preserving local homeomorphism onto the γ_2 orbit of $h(y)$. A *semiequivalence of flows* (or *flow semiequivalence*) is a surjective flow map. An *equivalence*

³ Already in [16], Hurewicz proved that a product of n one-dimensional compact metric spaces has dimension at least n .

⁴In our setting of compact metric spaces, a bijective homomorphism of flows must be a topological conjugacy.

of flows (or *flow equivalence*) is a semiequivalence of flows defined by a homeomorphism (in our compact metric setting, a bijective flow map).

By a *flow equivalence* of two homeomorphisms, we mean an equivalence of the suspension flows on their mapping tori.

For example, suppose flows on Y_1, Y_2 are built as flows under continuous positive functions φ_1, φ_2 with the same base homeomorphism $T : X \rightarrow X$. Then there is a flow equivalence $Y_1 \rightarrow Y_2$ which is an extension of the identity map between the bases $X \times \{0\}$.

Remark 2.10. Our choice of terminology for equivalence in Definitions 2.9 follows [14] and [29, Sec. 4.7]. It is well adapted to our topic of considering when two maps are flow equivalent (terminology from [26] and perhaps earlier): we naturally want to refer to a morphism by which two maps are flow equivalent as a flow equivalence. Caveat: various other terminologies have been used by different authors (e.g. [32, 18, 20]). For example, “topological conjugacy” in this paper is “ C^0 flow equivalence” in [20, Sec. 2.2], and “flow equivalence” in this paper is “ C^0 orbit equivalence” in [20, Sec. 2.2].

With the term “morphism” of flows committed by our use of isomorphism of flows, we end up using “flow map” for the corresponding notion related to flow equivalence.

Proposition 2.11. *Suppose $\pi : Y \rightarrow Y'$ is a flow map, with C, C' subsets of Y, Y' such that $C = \pi^{-1}(C')$. If π is surjective, then the following are equivalent.*

- (1) C is a cross section of Y .
- (2) C' is a cross section of Y' .

In general (i.e., if π is not assumed surjective), if C' is a cross section for Y' , then C is a cross section for Y .

Proof. (2) \implies (1) : Because C' is a cross section, π is a flow map, and $\pi^{-1}(C') = C$, the following hold: C is closed, every orbit hits C , and the return time $\tau_C(x) := \min\{t > 0 : \gamma_t(x) \in C\}$ is well defined for every $x \in C$. It remains to show that τ_C is bounded away from 0. Suppose not. Then there is a sequence $\{x_n\}$ in C such that $\tau_C(x_n) \rightarrow 0$. It follows from the compactness of C that there is a subsequence $\{x_{n_i}\}$ and an $x \in C$ such that $x_{n_i} \rightarrow x$. Let $\tau_{C'}$ be the return function for C' . Choose a $\delta > 0$ such that $\tau_{C'}(x') > \delta$ for every $x' \in C'$. Since the restriction of π to the γ orbit of x_{n_i} is an orientation-preserving local homeomorphism onto the γ' orbit of $\pi(x_{n_i})$, it follows that there is a $t_i \in (0, \tau_C(x_{n_i}))$ such that $\pi(\gamma(x_{n_i}, t_i)) = \gamma'(\pi(x_{n_i}), \delta)$. Then $t_i \rightarrow 0$, so $\gamma'(\pi(x_{n_i}), \delta) = \pi(\gamma(x_{n_i}, t_i)) \rightarrow \pi(x)$, but that cannot be the case since $\tau_{C'}(x') > \delta$ for every $x' \in C'$. Hence τ_C is bounded away from 0, and C is a cross section of Y .

(1) \implies (2) : Because C is a cross section, π is a flow map, and $\pi^{-1}(C') = C$, the following hold: C' is closed; every orbit hits C' ; the

return time $r'(x') := \min\{t > 0 : \gamma'_t(x') \in C'\}$ is well defined for each $x' \in C'$. It remains to show r' is bounded away from 0. Suppose not. Then there is a sequence $\{x'_n\}$ in C' such that $\tau_{C'}(x'_n) \rightarrow 0$. Choose for each n , an $x_n \in C$ such that $\pi(x_n) = x'_n$. It follows from the compactness of C that there is a subsequence $\{x_{n_i}\}$ and an $x \in C$ such that $x_{n_i} \rightarrow x$. Let τ_C be the return function for C . Choose $\delta > 0$ such that $\tau_C(y) > \delta$ for every $y \in C$. Since the restriction of π to the γ orbit of x_{n_i} is an orientation-preserving local homeomorphism onto the γ' orbit of $\pi(x_{n_i})$, it follows that there is a $t'_i \in (0, \tau_{C'}(x'_{n_i}))$ such that $\pi(\gamma(x_{n_i}, \delta)) = \gamma'(\pi(x_{n_i}), t'_i)$. Then $t'_i \rightarrow 0$. So $\pi(\gamma(x_{n_i}, \delta)) = \gamma'(\pi(x_{n_i}), t'_i) \rightarrow \pi(x)$. Thus $\pi(\gamma(x, \delta)) = \pi(x)$, from which it follows that $\gamma(x, \delta) \in C$, but that cannot be the case since $\tau_C(y) > \delta$ for every $y \in C$. Hence $\tau_{C'}$ is bounded away from 0, and C' is a cross section of Y' .

The nonsurjective case. If π is not surjective and C' is a cross section, then $C' \cap \pi(Y)$ is a cross section for the restriction of the Y' flow to $\pi(Y)$. Therefore the final claim follows from the case (2) \implies (1). \square

Definition 2.12. Suppose $T : X \rightarrow X$ is a homeomorphism of a compact zero dimensional metric space. A *discrete cross section* for T is a closed subset C of X with a continuous function $r : C \rightarrow \mathbb{N}$ such that $r(x) = \min\{k \in \mathbb{N} : T^k(x) \in C\}$ and $X = \{T^k(x) : x \in C, k \in \mathbb{N}\}$.

In Definition 2.12, the function r must be bounded and locally constant on C . Then X is the disjoint union of finitely many clopen sets of the form $T^i(C_j)$, $0 \leq i < j$, with $C_j = \{x \in C : r(x) = j\}$. Consequently, if K is a subset of a zero dimensional cross section C to a flow, then K is a cross section to the flow if and only if K is a discrete cross section for the discrete system (C, ρ_C) .

If $h : Y \rightarrow Y'$ is a flow map (perhaps an equivalence) and C' is a cross section for Y' , then $C = h^{-1}(C')$ is a cross section for Y and the restriction $h|_C$ defines a morphism of the return maps $\rho_C, \rho_{C'}$ (i.e., $h|_C$ is a continuous map from C into C' which intertwines ρ_C and $\rho_{C'}$). Conversely, if $\varphi : C \rightarrow C'$ is a morphism of return maps to cross sections C, C' for Y, Y' , then φ extends to a flow map $h : Y \rightarrow Y'$; in the case that $h^{-1}(C') = C$, we say this flow map is *induced by* φ . If h_1, h_2 are two flow maps induced by a morphism φ , then there is a flow equivalence $h_3 : Y \rightarrow Y$, which is an extension of the identity map on C and is isotopic to the identity in the group $\text{Homeo}_+(Y)$ of orientation-preserving orbit-preserving homeomorphisms on Y (Definitions 3.1), such that $h_1 = h_2 \circ h_3$.

3. ISOTOPY

We will now study when a flow equivalence mapping each orbit into itself is isotopic to the identity within the group of flow equivalences of a space Y to itself. We begin with some definitions.

Definitions 3.1. Suppose Y is a compact metric space with a fixed-point free continuous \mathbb{R} -action (a flow).

- (1) $\text{Homeo}_+(Y)$ is the group of homeomorphisms of Y which map flow orbits onto flow orbits, preserving the orientation given by the flow direction. This is the group of self-equivalences of the flow on Y .
- (2) $\text{Homeo}_+^{\text{orb}}(Y)$ is the group of homeomorphisms in $\text{Homeo}_+(Y)$ which map each flow orbit to itself.
- (3) $\text{Homeo}_+^{\text{iso}}(Y)$ is the subgroup of $\text{Homeo}_+^{\text{orb}}(Y)$ consisting of the homeomorphisms isotopic in $\text{Homeo}_+^{\text{orb}}(Y)$ to the identity.⁵
(In detail: $h \in \text{Homeo}_+^{\text{iso}}(Y)$ if there is a continuous map $H : Y \times [0, 1] \rightarrow Y$ such that, with $h_t(y) = H(y, t)$, each $h_t \in \text{Homeo}_+(Y)$, $h_1 = h$ and $h_0 = \text{Id}$.)

Equivalently, $\text{Homeo}_+^{\text{iso}}(Y)$ is the path component of the identity in $\text{Homeo}_+(Y)$ (with $\text{Homeo}_+(Y)$ topologized by a metric $\text{dist}(f, g) = \max\{d(f(y), g(y)) + d(f^{-1}(y), g^{-1}(y)) : y \in Y\}$, with d a metric on Y compatible with the topology).

Homeomorphisms φ_0, φ_1 in $\text{Homeo}_+^{\text{orb}}(Y)$ are isotopic in $\text{Homeo}_+^{\text{orb}}(Y)$ if and only if they are connected by a path φ_t in $\text{Homeo}_+^{\text{orb}}(Y)$ if and only if there is ψ in $\text{Homeo}_+^{\text{iso}}(Y)$ such that $\varphi_1 = \varphi_0 \circ \psi$.

When Y in Definition 3.1 is one-dimensional (i.e., has a zero dimensional cross section), the composants (path connected components) of Y are the flow orbits; so, an element h of $\text{Homeo}_+(Y)$ is isotopic in $\text{Homeo}_+(Y)$ to the identity if and only if it is an element of $\text{Homeo}_+^{\text{orb}}(Y)$ which is isotopic to the identity in $\text{Homeo}_+^{\text{orb}}(Y)$, i.e., if and only if it belongs to $\text{Homeo}_+^{\text{iso}}(Y)$.

Below, the image of a point y under the time t map of a flow γ is denoted $\gamma_t(y)$ or $\gamma(y, t)$. An ambient flow may be denoted γ without comment.

We begin with a standard example.

Example 3.2. Let T be the identity map on the unit circle. The suspension flow on the mapping torus of T can be presented as a flow on the 2-torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$, with $\gamma_t : [(x, y)] \mapsto [(x, y + t)]$. The (“Dehn twist”) toral automorphism $h : [(x, y)] \mapsto [(x, y + x)]$ maps each flow orbit to itself. But, h is not isotopic to the identity, because the homeomorphism h induces a nontrivial automorphism of the fundamental group of \mathbb{T}^2 . \square

In the next proposition, our main interest is in the case that $Y = \mathbf{S}X$ with X zero-dimensional. We use ρ to denote return map and τ to denote return time.

⁵By a theorem of Aliste-Prieto and Petite [2], $\text{Homeo}_+^{\text{iso}}(Y)$ is known to be a simple group.

Theorem 3.3. *Suppose γ is a flow on a compact metric space Y such that γ has no fixed point, and $h \in \text{Homeo}_+^{\text{orb}}(Y)$. Then the following are equivalent.*

- (1) *There is a continuous function $\beta : Y \rightarrow \mathbb{R}$ such that $h(y) = \gamma_{\beta(y)}(y)$, for all y in Y .*
- (2) *$h \in \text{Homeo}_+^{\text{iso}}(Y)$.*

Moreover, the following hold.

- (3) *Suppose C and D are cross sections for Y such that $h(C) = D$, and there is a continuous map $\beta : C \rightarrow \mathbb{R}$ such that the following hold:*
 - (a) *$h(y) = \gamma_{\beta(y)}(y)$, for all y in C .*
 - (b) *$\beta(\rho_C(y)) - \beta(y) = \tau_D(y)$.**Then (1) holds.*
- (4) *Given (1), for $0 \leq s \leq 1$ define $h_s : Y \rightarrow Y$ by the rule $h_s(y) = \gamma_{s\beta(y)} = \gamma(y, s\beta(y))$. Then $(h_s)_{0 \leq s \leq 1}$ is a path in $\text{Homeo}_+^{\text{iso}}(Y)$ from the identity to h .*

Proof. (1) \implies (2): We will prove this implication by proving (4). Because β is continuous, h_s is continuous. It remains to show for $0 < s < 1$ and $y \in Y$ that the restriction $h_s : \text{Orbit}(y) \rightarrow \text{Orbit}(y)$ is bijective and orientation preserving.

For $t \in \mathbb{R}$, $h_s(\gamma(y, t)) = \gamma(y, t + s\beta(\gamma(y, t)))$. Because $s\beta$ is bounded and continuous, it follows that $h_s : \text{Orbit}(y) \rightarrow \text{Orbit}(y)$ is surjective.

Suppose $\text{Orbit}(y)$ is not a circle. Considering $\gamma_r(y)$ and $\gamma_t(y)$, we see that h_s is orientation preserving and injective on $\text{Orbit}(y)$ if and only if

$$r < t \implies r + s\beta(\gamma_r(y)) < t + s\beta(\gamma_t(y))$$

which is equivalent to

$$(3.1) \quad r < t \implies s(\beta(\gamma_r(y)) - \beta(\gamma_t(y))) < t - r.$$

Because (3.1) holds for $s = 1$, it holds for $0 < s < 1$.

Now suppose $\text{Orbit}(y)$ is a circle, with p the smallest positive number such that $\gamma_p(y) = y$. The argument above, restricted to r, t such that $0 \leq r < t \leq p$, again shows h_s is orientation preserving and injective on $\text{Orbit}(y)$.

(2) \implies (1): We are given a continuous function $H : Y \times [0, 1] \rightarrow Y$, $(w, s) \mapsto h_s(w)$, with $h_1 = h$, $h_0 = \text{Id}$ and each $h_s \in \text{Homeo}_+^{\text{orb}}(Y)$. Pick $T > 0$ such that for all y the restriction of γ to $\{y\} \times [0, 4T]$ is injective. Appealing to uniform continuity of H , pick $\eta > 0$ such that for any $\psi = h_s \circ h_r^{-1}$ with $0 \leq r < s \leq \min(1, r + \eta)$ and for any y in Y there is $c(y, \psi)$ in $(-T, T)$ such that $\psi(y) = \gamma(y, c(y, \psi))$. By choice of T , the number $c(y, \psi)$ is unique, and it depends continuously on y . Pick an integer $n > 1/\eta$. Set $\psi_0 = \text{Id}$ and for $1 \leq i \leq n$ set $\psi_i = h_{i/n} \circ h_{(i-1)/n}^{-1}$.

On Y define

$$\beta(y) = \sum_{i=1}^n c(\psi_{i-1}(y), \psi_i).$$

Then β is continuous and $h_1(y) = \gamma_{\beta(y)}(y)$.

(3) : Let $c : C \times [0, \infty) \rightarrow [0, \infty)$ be the continuous function such that $c(x, 0) = 0$ and $h : \gamma(x, t) \rightarrow \gamma(h(x), c(x, t))$. Extend β to all of Y by defining β on $U := \{\gamma(x, t) : x \in C, 0 < t < \tau_C(x)\}$ to be $\beta : y = (x, t) \mapsto \beta(x) + c(x, t)$. By (a), β is continuous on the open set U and satisfies $h(y) = \gamma(y, \beta(y))$ everywhere. The condition (b) guarantees β remains continuous on C . \square

Example 3.4. For Y the mapping torus of a certain reducible shift of finite type, we exhibit h in $\text{Homeo}_+^{\text{orb}}(Y)$ which is not isotopic to the identity.

Let n be a positive integer, with $n > 1$. The matrix $A = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ defines an SFT X_A which consists of two fixed points and n connecting orbits. Let h be the homeomorphism of X_A which acts like the shift on one connecting orbit and equals the identity map on the other orbits. Then h is an automorphism of the shift σ_A and induces a homeomorphism $\tilde{h} : \text{SX}_A \rightarrow \text{SX}_A$. Here $h(y) = \gamma_{\beta(y)}$ with $\beta = 1$ on one connecting orbit and $\beta = 0$ elsewhere, and β is discontinuous at the two circles in SX_A . If β' is any function with $h = \gamma_{\beta'}$, then $\beta' - \beta$ must be zero on the connecting orbits, and β' cannot be continuous. Therefore h is not isotopic to the identity. \square

Example 3.5. We will now present an example of a mixing sofic shift X with an element h of $\text{Homeo}_+^{\text{orb}}(\text{SX})$ which is not isotopic to the identity.

Let X_A be the full two-shift on symbols a, b . Let X be the image of X_A under the factor map π which collapses the two fixed points a^∞, b^∞ of X_A to a single fixed point q and which collapses no other points. The quotient system is topologically conjugate to a mixing sofic shift; more precisely, a mixing near Markov shift [11]. Define a locally constant function g on X_A by the rule $g(x) = 0$ if $x_0x_1 = aa$, $g(x) = 1$ if $x_0x_1 = bb$, $g(x) = 1/2$ if $x_0 \neq x_1$. For $0 \leq t \leq 1$ and $x \in X_A$, define $h : \text{SX}_A \rightarrow \text{SX}_A$ by

$$h : [(x, t)] \mapsto [(x, t + (1 - t)g(x) + tg(\sigma_A(x)))].$$

The definitions at $[x, 1]$ and $[\sigma_A(x), 0]$ are consistent, and h is continuous. Then h is a self-equivalence of the flow on SX_A because for $0 \leq r < s \leq 1$, we have

$$\begin{aligned} (s + (1 - s)g(x) + sg(\sigma_A x)) - (r + (1 - r)g(x) + rg(\sigma_A x)) \\ = (s - r)(1 + g(\sigma_A x) - g(x)) > 0 \end{aligned}$$

because $|g(\sigma_A x) - g(x)| \leq 1/2$. The function β defined in the notation above by $[(x, t)] \mapsto (1-t)g(x) + tg(\sigma_A(x))$ is a continuous function on SX_A such that $h(y) = \gamma(y, \beta(y))$.

Now define $\bar{h} : SX \rightarrow SX$ by $\bar{h}(\pi(y)) = \pi(h(y))$. The map \bar{h} is well defined because $\pi(y) = \pi(y') \implies \pi(h(y)) = \pi(h(y'))$. It follows that $\bar{h} \in \text{Homeo}_+^{\text{orb}}(SX)$.

Suppose $\bar{h} \in \text{Homeo}_+^{\text{iso}}(SX)$. Then by Theorem 3.3 there is a continuous function $b : SX \rightarrow \mathbb{R}$ such that $\bar{h}(y) = \gamma(y, b(y))$ for all $y \in SX$. Define $\tilde{b} = b \circ \pi \in C(SX_A, \mathbb{R})$. Then $h(y) = \gamma(y, \tilde{b}(y))$ for all $y \in SX$. We must have $\tilde{b} = \beta$ on the set of aperiodic points; by density of this set and continuity, we must then have $\tilde{b} = \beta$ everywhere. But this is impossible, since $\beta(a^\infty) - \beta(b^\infty) \neq 0$. This contradiction shows that \bar{h} is not isotopic to the identity. \square

4. THE PARRY-SULLIVAN ARGUMENT

Theorem 4.2 below is a formulation and extension of the key argument (in our opinion) of the Parry-Sullivan paper [26]. That argument is the heart of the matter; still, Theorem 4.2 adds two features to the content of [26]. First, Parry and Sullivan considered only the invertible (flow equivalence) case of Theorem 4.2; Theorem 4.2 is not restricted to invertible maps (because our study of flow equivalence via canonical covers [8] forces us to consider noninvertible maps). Second, we include in Theorem 4.2 an explicit statement about isotopic triviality (not needed by Parry and Sullivan), with an eye to the mapping class group of a subshift, especially an irreducible shift of finite type. We also give a detailed proof of Theorem 4.2, to complement the succinct argument in [26], for those of us with a less direct pipeline to topological truth.

Another version of the Parry-Sullivan theorem (for irreducible Markov shifts, with a different argument and formulation) is given in [27, Section V.5].

We begin with a lemma. Condition (1) in Lemma 4.1 is not needed to prove Theorem 4.2. The group $\text{Homeo}_+^{\text{iso}}(Y)$ in the statement is defined in Definitions 3.1.

Lemma 4.1. *Suppose C, C' are cross sections for a flow γ on a one-dimensional space Y . Then there exists $q \in \text{Homeo}_+^{\text{iso}}(Y)$ such that the following hold.*

- (1) *There is a finite subset T of \mathbb{R} such that*

$$q(C') \subset \{\gamma(x, t) : x \in C, t \in T\}.$$
- (2) $q(C') \cap C = \emptyset$.

The homeomorphism q can be chosen arbitrarily close to the identity.

Proof. (2) follows from (1): given $q(C')$ satisfying (1) and any sufficiently small $\nu > 0$, $(\gamma_\nu \circ q)(C')$ is a cross section disjoint from C . So it remains to prove (1).

Let μ and μ' be the minimum return times to C and C' respectively under the flow. Suppose $0 < \epsilon < \min\{\mu/2, \mu'/2\}$.

Given $y \in C'$, pick v in C and $t \in \mathbb{R}$ such that $y = \gamma(v, t)$. Because $0 < \epsilon < \mu/2$, γ maps $C \times [t - \epsilon, t + \epsilon]$ homeomorphically to a neighborhood of y . Pick V' clopen in C' and δ in $(0, \epsilon)$ such that γ maps $V' \times (-\delta, \delta)$ homeomorphically to an open neighborhood of y contained in $\gamma(C \times (t - \epsilon, t + \epsilon))$.

By compactness, we may cover C' with finitely many such neighborhoods $U'_i := \gamma(V'_i \times (-\delta, \delta))$, $1 \leq i \leq N$. Suppose $i \neq j$ and $z \in U'_i \cap U'_j$. Then there are $x_i \in V'_i, x_j \in V'_j$ and $\{s_i, s_j\} \subset (-\delta, \delta)$ such that $z = \gamma(x_i, s_i) = \gamma(x_j, s_j)$. Then $\gamma(x_i, s_j - s_i) = x_j$ with $|s_j - s_i| < 2\delta < 2\epsilon < \mu'$, so $s_i = s_j$ and $x_i = x_j$. Therefore $U'_i \cap U'_j \subset \gamma((V'_i \cap V'_j) \times (-\delta, \delta))$. Replace each V'_i with the clopen set $V'_i \setminus \bigcup_{j>i} V'_j$. The open sets U'_i are now pairwise disjoint but their union still covers C' .

Choose for each i , a $t_i \in \mathbb{R}$ such that $U'_i \subset \gamma(C \times (t_i - \epsilon, t_i + \epsilon))$. Then, for each $y \in V'_i$ there is a unique $s(y) \in (-\epsilon, \epsilon)$ such that $y \in \gamma(C \times \{t_i - s(y)\})$. The function $y \mapsto s(y)$ is then continuous on V'_i . Given s in $(-\epsilon, \epsilon)$, let ℓ_s be the homeomorphism $[-\epsilon, \epsilon] \rightarrow [-\epsilon, \epsilon]$ which is the union of the increasing linear homeomorphisms $[-\epsilon, 0] \rightarrow [-\epsilon, s]$ and $[0, \epsilon] \rightarrow [s, \epsilon]$. With $y = \gamma(v, t)$, define $q : Y \rightarrow Y$ by

$$q(y) = \begin{cases} \gamma(v, \ell_{s(y)}(t)) & \text{if } y = \gamma(v, t) \text{ for } (v, t) \in V'_i \times (-\epsilon, \epsilon), \ 1 \leq i \leq N, \\ y & \text{otherwise.} \end{cases}$$

For each i , q maps $C' \cap U'_i$ into $\gamma(C \times \{t_i\})$, so (1) holds with $T = \{t_1, \dots, t_N\}$. The map q is a homeomorphism mapping flow orbits to themselves preserving orientation. On each U'_i , with $y = \gamma(v, t)$ as above, $q(y) = \gamma(y, \beta(y))$, with $\beta(y) = \ell_{s(y)}(t) - t$ continuous on U'_i . Define $\beta(y) = 0$ outside $\bigcup_i U'_i$. Then β is continuous on Y and $q(y) = \gamma(y, \beta(y))$. It then follows from Theorem 3.3 that $q \in \text{Homeo}_+^{\text{iso}}(Y)$.

Because $|\beta(y)| < 2\epsilon$ and $\epsilon > 0$ was arbitrarily small, q can be chosen arbitrarily close to the identity. \square

Theorem 4.2 ([26]). *Suppose Y, Y' are one-dimensional compact metric spaces with fixed point free flows γ, γ' for which C, C' are zero dimensional cross sections. Suppose $h : Y \rightarrow Y'$ is a flow map.*

Then there are discrete cross sections D, D' for $\rho_C, \rho_{C'}$, with $D \subset C$ and $D' \subset C'$ and $h^{-1}(D') = D$, such that h is a composition $h = h_2 \circ h_1$, where

- (1) $h_1 : Y \rightarrow Y$ lies in the group $\text{Homeo}_+^{\text{iso}}(Y)$,
- (2) $h_2 : Y \rightarrow Y'$ is induced by a morphism $(D, \rho_D) \rightarrow (D', \rho_{D'})$.

Proof. By Proposition 2.11, $h^{-1}(C')$ is a cross section for the flow on Y .

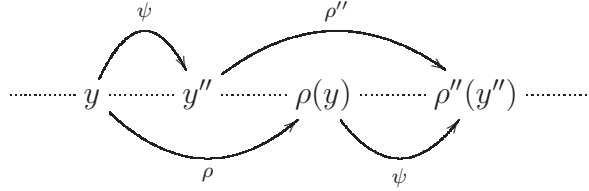
Case 1: $C \cap h^{-1}(C') = \emptyset$. Because C and $h^{-1}(C')$ are disjoint cross sections, there are continuous hitting time functions $C \cup h^{-1}(C') \rightarrow (0, \infty)$ defined by

$$\begin{aligned}\tau_1(y) &= \min\{t \in \mathbb{R} : t > 0, \gamma(y, t) \in C\}, \\ \tau_2(y) &= \min\{t \in \mathbb{R} : t > 0, \gamma(y, t) \in h^{-1}(C')\}.\end{aligned}$$

Define cross sections D, D'', D' for Y, Y, Y' (respectively) by

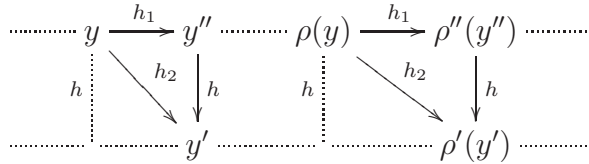
$$\begin{aligned}\{x \in C : \tau_2(x) < \tau_1(x)\} &= D \subset C, \\ \{\gamma(x, \tau_2(x)) : x \in D\} &= D'' \subset h^{-1}(C'), \\ h(D'') &= D' \subset C' .\end{aligned}$$

Let ρ, ρ'', ρ' be the return maps for the cross sections D, D'', D' under the flows γ, γ, γ' (respectively). Let $\psi : D \rightarrow D''$ be the homeomorphism $y \mapsto y''$ defined by $y'' = \gamma(y, \tau_2(y))$. Given y in D , the first four elements (in order along the flow) of $(D \cup D'') \cap \gamma(\{y\} \times [0, \infty))$ are $y, y'', \rho(y), \rho''(y'')$. Thus $\rho''(\psi(y)) = \psi(\rho(y))$



and ψ is a topological conjugacy $(D, \rho) \rightarrow (D'', \rho'')$.

Let $h_1 : Y \rightarrow Y$ be a flow equivalence induced by ψ^{-1} . Because τ_2 is continuous on D , it follows from Theorem 3.3 that $h_1 \in \text{Homeo}_+^{\text{iso}}(Y)$. Define $h_2 : Y \rightarrow Y'$ by $h_2 = h \circ h_1^{-1}$. The map h_2 restricts to a morphism $(D, \rho) \rightarrow (D', \rho')$



and the flow map h_2 is induced by this morphism.

Case 2: $C \cap h^{-1}(C') \neq \emptyset$. By Lemma 4.1 there is a $q \in \text{Homeo}_+^{\text{iso}}(Y)$ such that $q(C) \cap h^{-1}(C') = \emptyset$. The Case 1 argument then gives cross sections $D \subset q(C)$ and $D' \subset h^{-1}(C')$ such that $h = h_2 \circ h_1$ where $h_1 \in \text{Homeo}_+^{\text{iso}}(Y)$ and h_2 is induced by a morphism of return maps for D, D' . Then $q^{-1}(D)$ is a cross section; $q^{-1}(D) \subset C$; $h_2 \circ q$ is an equivalence induced by a morphism of the return maps for $q^{-1}(D)$ and D' ; $q^{-1} \circ h_1 \in \text{Homeo}_+^{\text{iso}}(Y)$; and $h = (h_2 \circ q) \circ (q^{-1} \circ h_1)$. \square

From the flow equivalence case of condition (2) in Theorem 4.2, Parry and Sullivan [26] and Bowen and Franks [5] derived invariants of flow equivalence for shifts of finite type which Franks [14] showed to be complete invariants for flow equivalence of nontrivial irreducible SFTs. For the Huang classification of general SFTs up to flow equivalence, see [6, 9]. For the classification up to flow equivalence for general SFTs with a free finite group action, see [7]. For partial results on the classification of sofic shifts up to flow equivalence, see [8].

5. FLOW CODES

If $\varphi : X \rightarrow Y$ is a continuous shift commuting map between subshifts, then φ is defined by a block code: a rule Φ defined on X -words of length $i + j + 1$ such that for all n and all x in X , $(\varphi(x))_n = \Phi(x_{n-i}x_{n-i+1} \cdots x_{n+j})$. We will introduce *flow codes* (and *word flow codes*) to get analogous invariant local codings for flow maps.

Let C be a discrete cross section for a subshift X . Given C , the *return time bisequence* of a point x in C is the bisequence $(r_n)_{n \in \mathbb{Z}}$ (with $r_n = r_n(x)$) such that

- (1) $\sigma^j(x) \in C$ if and only if $j = r_n$ for some n ,
- (2) $r_n < r_{n+1}$ for all n , and
- (3) $r_0 = 0$.

A *return word* is a word equal to $x[0, r_1(x))$ for some $x \in C$. Given $x \in C$ and $n \in \mathbb{Z}$, $W_n = W_n(x)$ denotes the return word $x[r_n, r_{n+1})$. In the context of a given C , when we write $x = \dots W_{-1}W_0W_1\dots$ below, we mean $x \in C$ and $W_n = W_n(x)$. Given $x \in C$ and $i \leq j$, the tuple $(W_n(x))_{n=i}^j$ is the $[i, j]$ return block of x . To know this return block is to know the word $W = W_i \cdots W_j$ together with its factorization as a concatenation of return words.

Definition 5.1. For a discrete cross section C of a subshift X , a *C word block code* is a function Φ , which for some positive integer M maps $[-M, M]$ return blocks occurring in X to words. A *word block code* is a *C word block code* for some C . The function φ from C into a subshift given by Φ is defined to map $x = (W_n)_{n \in \mathbb{Z}}$ to the concatenation $x' = (W'_n)_{n \in \mathbb{Z}}$, with $W'_n = \Phi(W_{n-M}, \dots, W_{n+M})$ and $x'[0, \infty) = W'_0W'_1\dots$. By an abuse of terminology, we will also call the function φ a word block code.

Let C and φ be as in Definition 5.1. Because C is clopen, there is a $\kappa \geq 0$ such that for all x in X , the word $x[-\kappa, \kappa]$ determines whether x is in C . If R is the maximum return time to C , it follows for all x in C that $x[-\kappa - MR, MR + \kappa]$ determines the return block $(W_{-M}(x), \dots, W_M(x))$. In particular, φ is continuous on C . Notice that φ is not required to be injective or surjective.

Definition 5.2. Let X, X' be subshifts and let C be a discrete cross section of X . A C word flow code is a flow map $h : \mathbf{S}X \rightarrow \mathbf{S}X'$ defined from a C word block code φ as follows: for each $x = (W_n)_{n \in \mathbb{Z}}$ and $\varphi(x) = x' = (W'_n)_{n \in \mathbb{Z}}$,

$$h : [(x, t)] \mapsto [(x', ct)], \quad 0 \leq t \leq |W_0|,$$

with $c = |W'_0|/|W_0|$. A word flow code is a C word flow code for some discrete cross section C .

Let $y = [x, 0]$ be a point in $\mathbf{S}X$ with x in C . We can visualize the word flow code map on the orbit of y by adding vertical bars to display the factorization into return words:

$$\begin{array}{ccccccc} y = \cdots & | & x_{-3}x_{-2}x_{-1} & | & x_0x_1 & | & x_2 & | & x_3x_4x_5 & | & \cdots \\ & & & & \downarrow & & & & & & \\ h(y) = \cdots & | & x'_{-2}x'_{-1} & | & x'_0x'_1x'_2 & | & x'_3x'_4 & | & x'_5 & | & \cdots \end{array}$$

The coordinates of x' covered by W'_n may grow arbitrarily far from the coordinates $[r_n, r_{n+1})$ of x covered by W_n . Nevertheless, $h : \mathbf{S}X \rightarrow \mathbf{S}X'$ is defined by patching together local rules.

Given φ a C word block code, let $X_{\varphi(C)}$ be the subshift which is the shift closure of $\{\varphi(x) : x \in C\}$. Then φ defines a word flow code $h : \mathbf{S}X \rightarrow \mathbf{S}X_{\varphi(C)}$. However, $\varphi(C)$ need not be a discrete cross section for $X_{\varphi(C)}$, even if φ is a block code, because in general $\varphi(C)$ need not be open as a subset of $X_{\varphi(C)}$ (Example 5.7). Even if $\varphi(C)$ is a discrete cross section, we would like to insist that $C = h^{-1}(h(C))$ so that the flow code defined from φ will be induced by a morphism of return maps, $(C, \rho_C) \rightarrow (\varphi(C), \rho_{\varphi(C)})$. So, we will refine the definition of word flow code.

To be completely explicit, suppose C, C' are discrete cross sections for subshifts X, X' . For W an X -word or X' word, we use notation $[W]$ to denote the word viewed as a symbol in an alphabet. For x in C , let η_C map $x = \cdots W_{-1}W_0W_1 \cdots$ to the bisequence $\cdots [W_{-1}][W_0][W_1] \cdots$. Then $\eta(C)$ is compact and invariant under the shift map σ , and $\eta_C : (C, \rho_C) \rightarrow (\eta_C(C), \sigma)$ is a topological conjugacy. Now suppose $\psi : (C, \rho_C) \rightarrow (C', \rho_{C'})$ is a morphism of discrete systems. Then the map $\eta_{C'} \circ \psi \circ \eta_C^{-1} : \eta_C(C) \rightarrow \eta_{C'}(C')$ is a block code, which is used to define a C word code Φ and from that a C word flow code $h : \mathbf{S}X \rightarrow \mathbf{S}X'$. This h is the word flow code induced by the morphism $\psi : (C, \rho_C) \rightarrow (C', \rho_{C'})$.

Definition 5.3. Let X, X' be subshifts with discrete cross sections C, C' . A (C, C') flow code is a flow map $h : \mathbf{S}X \rightarrow \mathbf{S}X'$ such that the following hold:

- (1) h is a word flow code defined by a morphism $(C, \rho_C) \rightarrow (C', \rho_{C'})$.
- (2) $h^{-1}(C') = C$.

A flow code is a (C, C') flow code for some (C, C') as above.

Example 5.4. Suppose a is a symbol from the alphabet $\mathcal{A}(X)$ of a subshift X and a' is not in $\mathcal{A}(X)$. Define a word code on X which copies each symbol except a , and maps a to aa' . This is a *symbol expansion*. Here $\varphi(X)$ is a discrete cross section in $X_{\varphi(X)}$, and the resulting $(X, \varphi(X))$ flow code is a flow equivalence. For example, for X the 2-shift on alphabet $\{a, b\}$ and X' the golden mean shift on alphabet $\{a, a', b\}$ (with $(b^*(aa')^*)^*$ its language), this symbol expansion gives a flow equivalence $\mathbf{S}X \rightarrow \mathbf{S}X'$.

In Theorem 5.5 below, for brevity we identify the base cross section $\{[(x, 0)] : x \in X\}$ of $\mathbf{S}X$ with X , and the return map with the shift map on X .

Theorem 5.5. *Suppose $h : \mathbf{S}X \rightarrow \mathbf{S}X'$ is a flow equivalence of mapping tori of subshifts. Then there is a flow code $h_2 : \mathbf{S}X \rightarrow \mathbf{S}X'$ and an $h_1 \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X)$ such that $h = h_2 \circ h_1$. The map h_1 has the form $h_1 : y \mapsto \gamma(y, \beta(y))$, with $\beta : Y \rightarrow \mathbb{R}$ continuous.*

Proof. As in Theorem 4.2, $h = h'_2 \circ h'_1$, with h'_2 induced by a topological conjugacy of the return maps to discrete cross sections C, C' in X, X' and $h'_1 \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X)$. We identify the base cross sections X, X' of the mapping tori with subshifts, with return maps given by the shift map. We may present these return maps as subshifts, with alphabets the respective return word sets. The topological conjugacy, as a block code with respect to these alphabets, is a word block code which induces a flow code h_2 such that $h'_2 = h_2 \circ \varphi$ for some $\varphi \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X)$. Let $h_1 = \varphi \circ h'_1$. Then $h_1 \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X)$ and $h = h_2 \circ h_1$. The form for h_1 follows from Theorem 3.3. \square

We could briefly summarize Theorem 5.5 by saying that every flow equivalence of subshifts is isotopic to one given by a flow code.

Flow codes are adapted to formulating coding arguments for flow equivalence of subshifts (analogous to block codes for shift-commuting maps between subshifts).

Remark 5.6. The fact that a flow equivalence of mapping tori of subshifts (when it exists) is isotopic to a nice one (given by a flow code) is reminiscent of the fact that a continuous equivalence of smooth flows on compact hyperbolic sets has a C^0 perturbation to a Hölder continuous equivalence [20, Theorem 19.1.5].

Finally we detail the example referred to earlier.

Example 5.7. Let (X, T) be the golden mean shift on symbols a_1, a_2, b , with language $((a_1 a_2)^* b^*)^*$, and let φ be the one block code with rule $a_1 \mapsto a, a_2 \mapsto a, b \mapsto b$. Then $C = \{x : x_0 = a_1 \text{ or } x_0 = b\}$ is a discrete cross section of (X, T) ; $\varphi(X) = X_{\varphi(C)}$; the fixed point a^∞ is in $\varphi(C)$; and $\varphi(C)$ contains no $\varphi(X)$ neighborhood of a^∞ (because for every positive integer n , $\varphi(C)$ contains no point x such that $x[0, 2n + 1] =$

$a^{2n+1}b$. Therefore $\varphi(C)$ is not open (alternately, note the return time to $\varphi(C)$ is not continuous). Therefore $\varphi(C)$ is not a discrete cross section for the subshift $X_{\varphi(C)}$.

6. FLOW EQUIVALENCE INDUCED BY CONJUGACY

We consider conditions on a flow equivalence of mapping tori which can force it to be induced by a topological conjugacy of their bases.

For completeness, we first recall an old, simple lemma (for which we do not know the original reference). Given an edge e in a directed graph, we let $\iota(e)$ be its initial vertex and $\tau(e)$ its terminal vertex. By a cycle of edges, we mean a string of edges $e_0 \dots e_k$ such that $\tau(e_i) = \iota(e_{i+1})$ for each i , and $\tau(e_k) = \iota(e_0)$.

Lemma 6.1. *Suppose f is a function from edges of an irreducible finite directed graph \mathcal{G} into an abelian group G , such that for every cycle of edges $e_0 \dots e_k$ in the graph, $\sum_{0 \leq i \leq k} f(e_i) = 0$. Then there is a function h from vertices into G such that for all e , $f(e) = h(\tau(e)) - h(\iota(e))$.*

Proof. Pick a vertex v_0 in the graph. Define $h(v_0) = 0$. Given a vertex v , by irreducibility there exists a path $e_0 \dots e_k$ with $\iota(e_0) = v_0$ and $\tau(e_k) = v$. Define $h(v) = \sum_{0 \leq i \leq k} f(e_i)$. By the zero-on-cycles assumption, $h(v)$ does not depend on the path (it must be the negative of the sum of h along any path from v_0 back to v). \square

Proposition 6.2. *Suppose f is a locally constant function from an irreducible SFT into an abelian group, and f sums to zero over each periodic orbit. Then there is a locally constant function b such that $f = b \circ \sigma_A - b$.*

Proof. After passing to a higher block presentation, without loss of generality one can assume the SFT is an edge shift and $f(x)$ depends only on the edge x_0 , $f(x) = f(x_0)$. Let h be as in Lemma 6.1 and define $b(x) = h(\iota(x_0))$. Then $g(x) = h(\sigma x) - h(x)$. \square

For a generalization of Proposition 6.2 to nonabelian groups (with a more subtle conclusion), see [25, Theorem 9.3] and [30]; for a remark on that conclusion, see [12, Remark 4.7]. Proposition 6.2 is a special case of a subshift version of the Livšic Theorem [20, Thm. 19.2.1].

The next result is another subshift version of a theorem for smooth hyperbolic flows (see [20, Theorem 19.2.8]).

Theorem 6.3. *Suppose there are flows on Y, Y' having (respectively) cross sections with return maps identified by topological conjugacy with irreducible SFTs (X_A, σ) and (X_B, σ_B) . Suppose $h : Y \rightarrow Y'$ is an equivalence of flows such that for each circle \mathcal{C} in SX_A , $|\mathcal{C} \cap X_A| = |h(\mathcal{C}) \cap X_B|$. Then h is isotopic to an equivalence induced by a topological conjugacy of the given SFTs, $X_A \rightarrow X_B$.*

Finally, for a point x with dense orbit in X_A , let $\tau \in \mathbb{R}$ be such that k takes $[(x, 0)]$ to $[(x', \tau)]$, with $x' \in X_B$ (τ is unique if X_A is not just a single finite orbit.) Then for every i in \mathbb{Z} , $\gamma_{-\tau} \circ j$ takes $[\sigma_A^i(x), 0]$ to

$[\sigma_B^i(x'), 0]$. By density of the orbit in X_A , $\gamma_{-\tau} \circ j$ takes $X_A \times \{0\}$ to $X_B \times \{0\}$, and therefore defines a topological conjugacy of σ_A and σ_B (given the obvious identification of $\sigma : X \rightarrow X$ with the return map to $X \times \{0\}$ under the suspension flow). \square

Theorem 6.4. *Suppose $h \in \text{Homeo}_+^{\text{orb}}(\mathbf{S}X)$, with X an irreducible SFT. Then $h \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X)$.*

Remark 6.5. Theorem 6.4 is also known to hold when X is a minimal subshift (see [2, Theorem 2.5] and its references).

Proof of Theorem 6.4. By Proposition 6.3, h is isotopic to an equivalence induced by an automorphism of an irreducible SFT fixing every periodic orbit. Such an automorphism can only be a power of the shift [10], so h is isotopic to the identity. (For an alternate proof of the corollary, take τ in the proof of Theorem 6.3 such that j maps $[(x, 0)]$ to itself.) \square

Example 6.6. Theorem 6.4 fails for reducible SFTs (Example 3.4) and mixing sofic shifts (Example 3.5). Theorem 6.3 also becomes false if the assumption “irreducible SFT” is replaced with “irreducible sofic” (or with “mixing sofic”). To see this, note that $h : \mathbf{S}X \rightarrow \mathbf{S}X$ in Example 3.5 takes each orbit of $\mathbf{S}X$ into itself. Suppose this h is induced by an automorphism U of X . As a factor map, π is conjugate to the Fischer cover of a sofic shift; therefore, there is a unique automorphism \tilde{U} of X_A which is a lift of U [22]. Because \tilde{U} must fix all periodic orbits of X_A , except perhaps the two fixed points, \tilde{U} must be a power of the shift [10]. Therefore U is a power of the shift, and therefore h is isotopic to the identity. But this contradicts the conclusion of Example 3.5.

Remark 6.7. We are considering fixed-point-free flows on a mapping torus $Y = \mathbf{S}X$, with γ_t the time t map of the flow. That X is a cross section to the flow means that there is a surjective local homeomorphism $\pi : X \times \mathbb{R} \rightarrow Y$ such that $\pi(x, t) = \gamma_t(x)$. In this context of considering isotopy of maps, one might hope that for a flow equivalence $h : \mathbf{S}X \rightarrow \mathbf{S}X'$, there might be a homeomorphism $\tilde{h} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ such that $\tilde{h} \circ \pi = \pi \circ h$. In general, no such lift exists.

Proposition 6.8. *Suppose for $i = 1, 2$ that $T_i : X_i \rightarrow X_i$ is a homeomorphism of a zero dimensional compact metric space. Let γ_i be the suspension flow on the mapping torus $\mathbf{S}X_i$. Let $\pi_i : X_i \times \mathbb{R} \rightarrow \mathbf{S}X_i$ be the map $(x, t) \mapsto \gamma_i(x, t)$. Suppose $h : \mathbf{S}X_1 \rightarrow \mathbf{S}X_2$ is a topological equivalence of the suspension flows.*

Then the following are equivalent.

- (1) *h is isotopic in $\text{Homeo}_+^{\text{orb}}(Y)$ to an equivalence of flows $Y_1 \rightarrow Y_2$ induced by a topological conjugacy of (X_1, T_1) and (X_2, T_2) .*

(2) *There is a homeomorphism $\tilde{h} : X_1 \times \mathbb{R} \rightarrow X_2 \times \mathbb{R}$ such that $h \circ \pi_1 = \pi_2 \circ \tilde{h}$.*

Proof. (1) \implies (2) : From the isotopy, there is a $j \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X_1)$ such that $h \circ j$ is a conjugacy of suspension flows sending the cross section $\pi_1(X_1 \times \{0\})$ onto $\pi_2(X_2 \times \{0\})$. Let $k : X_1 \rightarrow X_2$ be the homeomorphism such that for x in X_1 , $h \circ j : \pi_1(x, 0) \rightarrow \pi_2(k(x), 0)$. Then the map $X_1 \times \mathbb{R} \rightarrow X_2 \times \mathbb{R}$ defined by $(x, t) \mapsto (k(x), t)$ is a lift of $h \circ j$. It now suffices to check that there is a lift \tilde{j} of j . By Theorem 3.3, there is a continuous function $\beta : \mathbf{S}X_1 \rightarrow \mathbb{R}$ such that for all x in X_1 and $t \in \mathbb{R}$, $j : [(x, t)] \mapsto \gamma([(x, t)], \beta(x, t))$. Define $\tilde{j} : X_1 \times \mathbb{R} \rightarrow X_1 \times \mathbb{R}$ by $\tilde{j} : (x, t) \mapsto (x, t + \beta(\gamma_t(x)))$. Then \tilde{j} is continuous; $\tilde{j} \circ \pi_1 = \pi_1 \circ \tilde{j}$; for each $x \in X_1$, \tilde{j} is bijective on $x \times \mathbb{R}$; and \tilde{j} is a homeomorphism.

(2) \implies (1) : The given lift $\tilde{h} : X_1 \times \mathbb{R} \rightarrow X_2 \times \mathbb{R}$ has the form $\tilde{h} : (x, t) \mapsto (\bar{h}(x), t + \beta(x, t))$, with β continuous and $\bar{h} : X_1 \rightarrow X_2$ a homeomorphism. Because π_1 is a local homeomorphism and $h \circ \pi_1 = \pi_2 \circ \tilde{h}$, it follows that β is the lift of a continuous function (also denoted β) on $\mathbf{S}X_1$: for all $x \in X_1$ and all t , $h : \gamma(x, t) \mapsto [(\bar{h}(x), t + \beta(x, t))]$. Define $j : \mathbf{S}X_1 \rightarrow \mathbf{S}X_1$ by $y \mapsto \gamma(y, \beta(y))$. Because \tilde{h} is orientation preserving, for every x in X we have

$$s < t \implies s + \beta(x, s) < t + \beta(x, t).$$

Consequently $j \in \text{Homeo}_+(\mathbf{S}X_1)$, and by Theorem 3.3 it follows that $j \in \text{Homeo}_+^{\text{iso}}(\mathbf{S}X_1)$. Therefore h is isotopic in $\text{Homeo}_+^{\text{orb}}(Y)$ to $h \circ j^{-1}$, which lifts to the map $(x, t) \mapsto (\bar{h}(x), t)$. Therefore the equivalence $h \circ j^{-1}$ is a conjugacy of flows induced by the topological conjugacy $(X_1, T_1) \rightarrow (X_2, T_2)$ defined by \bar{h} . \square

Remark 6.9. If in Proposition 6.8(2) we only require \tilde{h} to be continuous, then \tilde{h} can easily be constructed: first define on $X_A \times \{0\}$, then extend. This \tilde{h} need not be surjective and need not be injective (even if $X_1 = X_2$). However, $\pi_2 \circ \tilde{h}$ will be surjective.

7. EXTENDING EQUIVALENCES AND CROSS SECTIONS

We finish with a pair of extension results for flows with zero dimensional cross sections. The definition of $\text{Homeo}_+^{\text{iso}}(Y)$ is in Definition 3.1.

Proposition 7.1. *Suppose a flow on a compact metric space Y has a zero dimensional cross section. Suppose Y' in Y is the domain of a subflow of Y and $\chi \in \text{Homeo}_+^{\text{iso}}(Y')$. Then χ extends to $\tilde{\chi} : Y \rightarrow Y$ such that $\tilde{\chi} \in \text{Homeo}_+^{\text{iso}}(Y)$.*

Proof. Without loss of generality, we assume Y is the mapping torus $\mathbf{S}X$ of a homeomorphism $T : X \rightarrow X$ of a zero dimensional compact metric space X .

By Theorem 3.3, there is a continuous map $\varphi : Y' \rightarrow \mathbb{R}$ such that

$$\chi(y) = \gamma_{\varphi(y)}(y)$$

for $y \in Y'$. By Theorem 3.3, it suffices to extend φ to a continuous map $\tilde{\varphi} : SX \rightarrow \mathbb{R}$ such that the function $\tilde{\chi} : x \mapsto \gamma_{\tilde{\varphi}(x)}(x)$ maps each orbit of SX to itself by an orientation preserving homeomorphism.

Let $X' = \{x \in X : [x, 0] \in Y'\}$. Then X' is T -invariant and $Y' = \{[y, t] : x \in X', t \in \mathbb{R}\}$. Since χ is orientation preserving, we have for all x in X' that

$$1 + \varphi([x, 1]) - \varphi([x, 0]) > 0.$$

By the continuity of φ and compactness of Y' , there is a $\mu > 0$ such that

$$\mu = \min\{1 + \varphi([x, 1]) - \varphi([x, 0]) : x \in X'\}.$$

Let \mathcal{P}_n , $n \in \mathbb{N}$, be a sequence of nested finite partitions of X into clopen sets, with $\text{diameter}(C) < 1/n$ for all C in \mathcal{P}_n . Let $\mathcal{Q}_n = \{C \in \mathcal{P}_n : C \cap X' \neq \emptyset\}$. Pick N such that for any pair x, z from X' ,

$$\text{dist}(x, z) < 1/N \implies |\varphi([x, 0]) - \varphi([z, 0])| < \mu/3.$$

From each C in \mathcal{Q}_n with $n \geq N$, pick a point $z_{n,C}$ in X' . Let Q be the clopen set in X which is the union of the C in \mathcal{Q}_N . Define continuous maps $\pi : Q \rightarrow X'$ and $\alpha : Q \rightarrow \mathbb{R}$ by

$$\begin{aligned} \pi(x) &= \begin{cases} x & \text{if } x \in X', \\ z_{n,C} & \text{if } x \in C \in \mathcal{Q}_n \text{ and } x \notin \bigcup_{C \in \mathcal{Q}_{n+1}} C, \end{cases} \\ \alpha(x) &= \varphi([\pi x, 0]). \end{aligned}$$

Note, if both x and $T(x)$ are in Q , then

$$\begin{aligned} 1 + \alpha(T(x)) - \alpha(x) &= 1 + \varphi(\pi(T(x))) - \varphi(\pi(x)) \\ &= 1 + (\varphi(\pi(T(x))) - \varphi(T(x))) + (\varphi(T(x)) - \varphi(x)) + (\varphi(x) - \varphi(\pi(x))) \\ &> (1 + \varphi(T(x)) - \varphi(x)) - 2\frac{\mu}{3} > \frac{\mu}{3} \end{aligned}$$

and therefore

$$(7.1) \quad 1 + \alpha(T(x)) - \alpha(x) > 0.$$

Choose an integer M such that

$$(7.2) \quad M > \max\{|\alpha(x)| : x \in Q\}.$$

Define clopen subsets of X ,

$$\begin{aligned} Q^{(M)} &= \bigcap_{-M \leq n \leq M} T^n Q \\ W &= \{x \in X : |n| < M \implies T^n(x) \notin Q^{(M)}\} \\ V &= W \cup Q^{(M)}. \end{aligned}$$

Extend α to a continuous function on V by setting $\alpha(w) = 0$ for $w \in W$.

The set $C := \{[v, 0] : v \in V\}$ is a cross section for SX . For $v \in V$, let $\tau_C(v)$ be the return time of $[v, 0]$ to C under the flow, and let $R_C(v)$ be the unique element $v' \in V$ for which $[v', 0] = \gamma_{\tau_C(v)}([v, 0])$. We make the Claim: for all v in V ,

$$(7.3) \quad \tau_C(v) + \alpha(R_C(v)) - \alpha(v) > 0.$$

We check the claim by four cases.

Case I: v and $R_C(v)$ belong to $Q^{(M)}$.

Here $\tau_C(v) + \alpha(R_C(v)) - \alpha(v) = 1 + \alpha(T(v)) - \alpha(v) > 0$ by (7.3).

Case II: $v \in W$ and $R_C(v) \in W$.

Here $\tau_C(v) + \alpha(R_C(v)) - \alpha(v) = 1 + 0 - 0 > 0$.

Case III: $v \in Q^{(M)}$ and $R_C(v) \in W$.

This is the case that $v \in Q^{(M)}$; $T^M(v) \notin Q$; and $0 < n < M \implies T^n(v) \in Q$. Then $\tau_C(v) + \alpha(R_C(v)) - \alpha(v) = M + 0 - \alpha(v)$, which is positive by (7.2).

Case IV: $v \in W$ and $R_C(v) \in Q^{(M)}$.

The argument is very similar to that of Case III. The Claim is proved.

From here, the notation $[v, s]$ refers to $v \in V$ and $0 \leq s \leq \tau_C(v)$. We define $\tilde{\chi} : Y \rightarrow Y$ by setting

$$\tilde{\chi} : [v, s] \mapsto [v, \alpha(v) + \frac{s}{\tau_C(v)}(\tau_C(v) + \alpha(R_C(v)) - \alpha(v))].$$

The definition is consistent because $[v, \tau_C(v)] \mapsto [[v, \tau_C(v)], \alpha(R_C(v))]$ is in agreement with $[R_C(v), 0] \mapsto [R_C(v), \alpha(R_C(v))]$. The map $\tilde{\chi}$ is continuous. It then follows from the Claim that $\tilde{\chi}$ sends each orbit to itself by a map which is piecewise (hence globally) an orientation preserving homeomorphism. Finally, define

$$\tilde{\varphi} : [v, s] \mapsto \left(\frac{s}{\tau_C(v)} \right) \alpha(R_C(v)) + \left(1 - \frac{s}{\tau_C(v)} \right) \alpha(v).$$

Then $\tilde{\varphi}$ is continuous on Y and

$$\tilde{\chi} : [v, s] \mapsto \gamma([v, s], \tilde{\varphi}([v, s])).$$

This completes the proof that $\tilde{\chi} \in \text{Homeo}_+^{\text{iso}}(Y)$. \square

We do not know if the assumption of a zero dimensional cross section in Proposition 7.1 is necessary.

Proposition 7.2. *Suppose γ is a flow on a one dimensional compact metric space Y ; Y' is compact and invariant in Y ; and C' is a cross section for the subflow on Y' .*

Then there is a cross section C for Y such that $C \cap Y' = C'$.

Proof. Let X be a zero dimensional cross section for the flow on the one-dimensional space Y (Proposition 2.8). Then $X' = Y' \cap X$ is a cross section for the subflow on Y' . For simplicity and without loss of generality, suppose Y is the mapping torus SX .

By Lemma 4.1, there exists a finite subset T of \mathbb{R} and $q \in \text{Homeo}_+^{\text{iso}}(Y)$ such that $q(C'') \subset \{\gamma(x, t) : x \in X', t \in T\}$. For $t \in T$, define $D'_t = \{x \in X' : \gamma(x, t) \in q(C'')\}$ and $D' = \cup_t D'_t$. Each D'_t is clopen in X' . Choose clopen subsets D_t of X such that $D_t \cap X' = D'_t$; then

$$Y' \cap (\cup_t \gamma(D_t \times \{t\})) = Y' \cap q(C'').$$

Let $D_X = \cup_{t \in T} D_t$, a clopen subset of X . Let R be the maximum return time under γ' to D' and set $E = X \cap \gamma(D_X \times [0, R])$. Then E is clopen in X and $Y' \cap E = \emptyset$. Define

$$D = \gamma(E \times \{0\}) \cup (\cup_t \gamma(D_t \times \{t\})).$$

Then D is a cross section for the flow on Y and $D \cap Y' = q(C'')$.

Let $\chi = q^{-1} \in \text{Homeo}_+^{\text{iso}}(Y')$. By Proposition 7.1, χ extends to $\tilde{\chi}$ in $\text{Homeo}_+^{\text{iso}}(Y)$. Now $\tilde{\chi}(D)$ is a cross section for the flow on Y and $Y' \cap \tilde{\chi}(D) = \tilde{\chi}(Y' \cap D) = \tilde{\chi}(q(C'')) = C''$. \square

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